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Irreversible Investment of the Risk- and Uncertainty-averse DM under k-Ignorance: The Role of BSDE *

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In this paper, the approach of BSDE will be employed to study the irreversible investment problem under k-ignorance when the DM is risk- and uncertainty-averse. For the case of logarithmic utility, we work out the explicit solutions of the value of the utilized patent, the value of the unutilized patent, and the value of the reservation profit. Furthermore, in view of numerical method, the effects of the risk and the uncertainty on the above three parameters are analyzed. All the comparative static results are consistent with our intuition.

Key Words: Irreversible investment; k-ignorance; Risk; Uncertainty; Backward stochastic differential equation (BSDE in short); Conditional g-expectation.

JEL Classification Numbers: C61, G11, D81.

1. INTRODUCTION

Usually, the investment decision of any firm typically involves three features. The first one is that the market conditions are uncertain in the future. Second, the cost of the investment is sunk and thus the investment is irreversible. Third, investment opportunity does not vanish at once, therefore when to invest becomes a critical decision. Recently, this kind of irreversible investment problem have attracted considerable attentions, See McDonald and Siegel (1986) and Dixit and Pindyck (1994). They used the approach of financial option pricing theory, which offered an elegant way to obtain the optimal investment strategy. In these traditional re-

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search papers, they also assumed that the uncertainty environment could be characterized by a certain probability measure over states of nature, which implied that decision maker of a firm (DM in short) was perfectly certain that market condition in the future was governed by this particular probability measure. However, this assumption may not be reasonable: the DM may not be so sure about future uncertainty. It may think other probability measures are also likely and have no idea of relative "plausibility" of these measures. A natural question is: How to describe this phenomena?

"Knightian uncertainty" or "ambiguity" in alternatives, introduced by Knight (1921) and Keynes (1921, 1936), will perfectly characterize the above uncertainty phenomena. In particular, uncertainty that is reducible to a single probability measure with known parameters is referred to as "risk". Epstein (1999) gave the rigorous definitions of risk and uncertainty, he also stated the definition of uncertainty aversion and some interesting results. In reality, the DM cannot completely assess the coming uncertainty, so it may face Knightian uncertainty in evaluating its investment. As a rule, Knightian uncertainty is often characterized by a set of probability measures, which also satisfy some additional restrictions.

Nishimura and Ozaki (2007), Schröder (2006) and Miao and Wang (2009) involved the Knightian uncertainty into the irreversible investment problem, the first two papers considered the Knightian uncertainty of "k-ignorance" in continuous time, the third paper considered the case of multiple-priors in discrete time. "k-ignorance", introduced by Chen and Epstein (2002), means that the set of probability measures deviate from $\mathbb{P}$ is not large in the sense that the element in it is absolutely continuous to the original probability measure $\mathbb{P}$ and the corresponding density generator's move is confined in a range, $[-k, k]$, where $k > 0$ can be described as a degree of this Knightian uncertainty. To highlight the effect of the uncertainty, they all assume that the DM is risk neutral, i.e., the utility function is linear. Some appealing theoretical and comparative static results are obtained.

However, in reality, the DM may evaluate the profit by a utility function, that is, the DM is risk averse. This means that it is essential to include some individual risk aversing parameters into the decision model for capturing the DM's attitude towards the risk he faces. It is widely known that the concavity of the utility function implies the risk-aversion of the decision maker. Thus, in this paper, we assume that the DM evaluates the profit of irreversible investment by a concave utility function. That is, we study the problem of the irreversible investment of risk- and uncertainty-averse DM under k-ignorance. When the DM evaluates his/her position, he/she will use a probability corresponding to the "worst" scenario, which means that the DM is uncertainty aversion. For this stream of literatures, the reader may refer to Gilboa and Schmeidler (1989), Gilboa (1987) and so on.

BSDE, introduced by Pardox and Peng (1990), plays an important role in many fields, such as stochastic control, mathematical finance, economics and so on. Via the solution of BSDE, Peng (1997) introduced the notions of g-expectation and conditional g-expectation, where they also proved that the conditional g-expectation preserved all properties of the classical conditional expectation except the linearity. Some applications of conditional g-expectation in economics and mathematical finance can be founded in Chen and Epstein (2002), El Karoui et al. (1997), Chen and Kulperger (2006) and references there in.

Under the framework of k-ignorance, Chen and Epstein (2002) proved that the infimum of the conditional expectation could be represented as a conditional g-expectation, which implied that it could be represented as the solution of BSDE. Chen and Kulperger (2006) defined this as the martingale representation theorem of the minimum conditional expectation. With the help of some results of BSDE, Chen and Kulperger (2006) also argued that the infimum of the conditional expectation can be obtained under k-ignorance for some special random variables. Motivated by this, in this paper we will use the results of BSDE to study the irreversible investment problem of risk- and uncertainty-averse DM under k-ignorance. The use of BSDE has the following three advantages: (i) The Min expectation induced by g-expectation is dynamic consistent; (ii) the closed form solutions of Maxmin expected utility can be easily obtained for geometric Brownian motion profit process; (iii) last but not least, we may consider the general profit flow process.

This paper proceeds as follows. In section 2, we list some preliminary results on BSDE and conditional g-expectation. The problem formulation and the general results are presented in Section 3. When the utility function is logarithmic and the profit flow follows geometric Brownian motion, we get the explicit solutions of the value of the utilized value, the value of the unutilized value and the value of the reservation profit value. These results are stated in Section 4. Some comparative results are also involved in this section. Section 5 is a brief conclusion. All the derivation and proofs are relegated in the Appendix.

2. BSDE AND G-EXPECTATION

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtrated probability space. Suppose that $(\mathcal{F}_t)_{t\geq 0}$ is the natural σ-filtration generated by the standard one-dimensional Brownian motion $(B_t)_{t\geq 0}$, that is,

$$\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}.$$

Let $T > 0, \mathcal{F}_T = \mathcal{F}$ and $g = g(t, y, z) : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ be a function satisfying

$(H1)$ $\forall (y, z) \in \mathbb{R} \times \mathbb{R}, g(t, y, z)$ is continuous in t and $\int_0^T g^2(t, 0, 0)dt < \infty$;

$(H2)$ g is uniformly Lipschtiz continuous in (y, z), that is, there exists a constant $c > 0$ such that $\forall y_1, y_2, z_1, z_2 \in \mathbb{R}$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq c(|y_1 - y_2| + |z_1 - z_2|);$$

$(H3)$ $g(t, y, 0) = 0, \ \forall (t, y) \in \mathbb{R} \times [0, T]$.

At the beginning of this section, we will list the existence and uniqueness theorem of the solution of BSDE and some properties of conditional g-expectation.

Let $\mathcal{M}(0, T, \mathbb{R})$ be the set of all square integrable $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted process $\{\nu_t\}$ with

$$\mathbb{E} \int_0^T |\nu_t|^2 dt < \infty.$$

For each $t \in [0, T]$, let $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the set of all $\mathcal{F}_t$-measurable random variables in $\mathcal{M}(0, T, \mathbb{R})$, Pardoux and Peng (1990) considered the following BSDE:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dB_s, 0 \leq t \leq T, \tag{1}$$

and showed the following result:

Lemma 1. *Suppose that g satisfies $(H1) - (H2)$ and $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then BSDE 1 has a unique solution $(y, z) \in \mathcal{M}(0, T, \mathbb{R}) \times \mathcal{M}(0, T, \mathbb{R})$.*

Via the solution of BSDE, Peng (1997) introduced the concept of g-expectation.

Definition 2.1. Suppose that g satisfies $(H1) - (H3)$. Given $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, let (y, z) be the solution of BSDE 1. We denote g-expectation of ξ by $\mathbb{E}c_g[\xi]$ and define it as

$$\mathbb{E}c_g[\xi] := y_0.$$

From the definition of g-expectation, Peng (1997) also introduced the concept of conditional g-expectation:

Lemma 2. *For any* $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, *there exists a unique* $\eta \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

$$\mathbb{E}c_g[I_A\xi] = \mathbb{E}c_g[I_A\eta], \ \forall A \in \mathcal{F}_t.$$

We call η *the conditional* g*-expectation of* ξ *and write* η *as* $\mathbb{E}c_g[\xi|\mathcal{F}_t]$. *Of course, such conditional expectations can be defined only for sub* σ*-algebra which appears in the filtration* $\{\mathcal{F}_t\}$. *Furthermore,* $\mathbb{E}c_g[\xi|\mathcal{F}_t]$ *is the value of the solution* $\{y_t\}$ *of BSDE 1 at time* t. *That is,*

$$\mathbb{E}c_g[\xi|\mathcal{F}_t] = y_t.$$

The conditional g-expectation preserves many of the properties of classical conditional mathematical expectation. However, it doesn't preserve linearity. See, for example, Peng (1997) for details.

Lemma 3.
(i) *For any constant* c, $\mathbb{E}c_g[c|\mathcal{F}_t] = c$;
(ii) *If* ξ *is* $\mathcal{F}_t$*-measurable, then* $\mathbb{E}c_g[\xi|\mathcal{F}_t] = \xi$;
(iii) *For any* $t, s \in [0, T]$,

$$\mathbb{E}c_g[\mathbb{E}c_g[\xi|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}c_g[\xi|\mathcal{F}_{t\wedge s}];$$

(iv) *If* $\xi_1 \geq \xi_2$, *then* $\mathbb{E}c_g[\xi_1|\mathcal{F}_t] \geq \mathbb{E}c_g[\xi_2|\mathcal{F}_t]$.

Remark 2.1.
1. g-expectation and conditional g-expectation depend on the choice of the function g, if g is nonlinear, then conditional g-expectation is usually also nonlinear.
2. If $g \equiv 0$, setting conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$ on both sides of BSDE 1 yields $y_t = \mathbb{E}c_g[\xi|\mathcal{F}_t] = \mathbb{E}[\xi|\mathcal{F}_t], y_0 = \mathbb{E}c_g[\xi] = \mathbb{E}[\xi]$. This implies another explanation for mathematical expectation: Within the particular framework of a Brownian filtration, conditional mathematical expectations with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ are the solution of a simple BSDE and mathematical expectation is the value of this solution at time $t = 0$.

In the discussion that follows, we are in position to describe k-ignorance, and related it to conditional g-expectation.

Define the following set $\mathcal{P}$ of probability measures [1],

$$\mathcal{P} = \left\{ \mathbb{Q}^\theta : \frac{d\mathbb{Q}^\theta}{d\mathbb{P}} = \exp\left\{ -\frac{1}{2}\int_0^T |\theta_s|^2 ds - \int_0^T \theta_s dB_s \right\}, \sup_{0\le s\le T} |\theta_s| \le k \right\}, k > 0,$$

where the process θ satisfies the Novikov's conditions, i.e. $\mathbb{E}[\exp\{\frac{1}{2}\int_0^T \theta_s^2 ds\}] < \infty$. For ease of expression, we also introduce the following notation

$$Y_t := \inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t].$$

To construct the relationship between k-ignorance and conditional g-expectation, we state two lemmas on $\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t]$ in order.

The first one is a special case of Theorem 2.2 in Chen and Epstein (2002). Chen and Kulperger (2006) defined this as the martingale representation theorem of minimum conditional expectation. In this lemma, the set of probability measures is a little different from Chen and Kulperger (2006), so the brief proof of this lemma is presented in the Appendix.

Lemma 4. *Given $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then Y_t has the following representation: there exists an adapted process $\{z_t\}$ such that (Y_t, z_t) is the solution of following BSDE*

$$Y_t = \xi - \int_t^T k|z_s|ds - \int_t^T z_s dB_s, 0 \le t \le T. \tag{2}$$

Remark 2.2. According to Lemma 4, we conclude that

$$\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t] = Y_t = \mathbb{E}c_g[\xi|\mathcal{F}_t],$$

where $g = -k|z|$, that is, $\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t]$ is a special kind of conditional g-expectation. Therefore, $\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t]$ satisfies the properties listed in Lemma 3.

Suppose that b and $\sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous in (t,x) and Lipschitz continuous in x. Let $\{X_t\}$ be the solution of the stochastic

[1] This set of probability measures are called k-ignorance. In this case, every element in $\mathcal{P}$ is absolutely continuous to $\mathbb{P}$, and the generators bounded by k, which can be seen as a measure of Knightian uncertainty, i.e., the uncertainty increases with the increase of k.

differential equation (SDE) below,

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad 0 \le t \le T, \tag{3}$$

which has a unique solution such that $X_T \in L^2(\Omega, \mathcal{F}, P)$.

The second lemma, due to Chen and Kulperger (2006), is crucial for the proof of the main result in this paper. The proof of this lemma is complex, technical and closely connected with the theory of BSDE, we omit it here. For the interested reader, ref. Chen and Kulperger (2006).

LEMMA 5. *Let Φ be an increase function such that $\Phi(X_T) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, if $\sigma(t, x) > 0$ for all $t \in [0, T]$ and $x \in \mathbb{R}$, then*

$$\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\Phi(X_T)|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}^k}[\Phi(X_T)|\mathcal{F}_t],$$

where $\frac{d\mathbb{Q}^k}{d\mathbb{P}} = \exp\{-\frac{1}{2}k^2T - kB_T\}$. In particular,

$$\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\Phi(B_T)|\mathcal{F}_t] = \mathbb{E}[\Phi(B_T - kT)|\mathcal{F}_t],$$

To our knowledge, this is the first time to use the approach of BSDE in studying irreversible investment problem. Using this method, there are three **advantages**: (i) According to Lemma 4, we needn't discuss the dynamic consistency of $\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t]$ because it is a special kind of conditional g-expectation; (ii) In view of 3 and Lemma 5, we may generalize the profit process to the form of 3; (iii) With the help of Lemma 5, we may consider the DM that is not only uncertainty averse but also risk averse.

3. PROBLEM FORMULATION AND GENERAL RESULTS

Let $b : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous in (t, x) and Lipschitz continuous in x. Without loss of generality [2], we assume that $\sigma(t, x) \ge 0$ for any $t \in [0, T]$ and $x \in \mathbb{R}$, and $\sigma(t, x) \ne 0$ to exclude a deterministic case. Here, suppose that the operating profit from the utilized patent is a real-valued stochastic process $(\pi_t)_{0 \le t \le T}$ that is generated by the following SDE

$$d\pi_t \;=\; b(t, \pi_t)dt + \sigma(t, \pi_t)dB_t, \pi_0 > 0. \tag{4}$$

[2] If necessary, take $(-B_t)$ instead of (B_t) in the following.

In the discussion that follows, we assume that the DM is risk- and uncertainty-averse with utility function $U : \mathbb{R} \to \mathbb{R}$, which is continuously differentiable, increasing and concave. In this case, if the profit flow follows the SDE 4 and the uncertainty averse DM evaluate the profit by a utility function, then the value at time t of the utilized patent with expiration time T is

$$W(\pi_t, t) = \inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\int_t^T e^{-\rho(s-t)} U(\pi_s) ds | \mathcal{F}_t] \tag{5}$$

where $\rho > 0$ [3] is the DM's discount rate. The infimum operator reflects the DM's uncertainty aversion, and the utility function reflects the DM's risk aversion.

In view of Lemma 5 and Fubini theorem of conditional expectation, we may get the following general formula. The proof is relegated to Appendix.

Proposition 1. *Suppose that the DM is risk- and uncertainty-averse under k-ignorance. Then, the value of the utilized patent in 5 is given by*

$$W(\pi_t, t) \;=\; \int_t^T e^{-\rho(s-t)} \mathbb{E}_{\mathbb{Q}^k}[U(\pi_s)|\mathcal{F}_t] ds. \tag{6}$$

Now, motivated by Nishimura and Ozaki (2007), we are in position to formulate the investment problem of the DM as an optimal stopping problem, and relate the investment problem to the value of the utilized patent described above.

To use the patent, the DM must invest I to building a factory. And the return (π_t) of this cost follows the SDE 4. Possessing the patent, as an investment opportunity, the DM's task is to contemplates when to invest is the optimal invest time. Then, at time t, the DM faces the optimal stopping problem of maximizing

$$\min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\int_\tau^T e^{-\rho(s-t)} U(\pi_s) ds - e^{-\rho(\tau-t)} I | \mathcal{F}_t]$$

by choosing an $\mathcal{F}_t$-stopping time $\tau \in [t, T]$, when the investment decision is to be made. The maximum of this problem is denoted by V_t :

$$V_t = \max_{\tau \geq t} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\int_\tau^T e^{-\rho(s-t)} U(\pi_s) ds - e^{-\rho(\tau-t)} I | \mathcal{F}_t]. \tag{7}$$

[3]Furthermore, we assume $\rho > \mu$, otherwise, the optimal solution will possibly not exist. We will explain this later.

Then, V_t is the value of investment opportunity.

Now, we can split the above optimal stopping problem into the two options available to the DM: invest now (at time t) or wait for a short time interval, dt, and reconsider whether to invest or not after that (at time $t + dt$). Then, by the same idea of Nishimura and Ozaki (2007), we may get the following result. For the convenience of the reader, we state its proof in the Appendix.

PROPOSITION 2. *V_t solves the following version of the Hamilton-Jacobi-Bellman equation:*

$$V_t = \max\{W_t - I, \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[dV_t|\mathcal{F}_t] + V_t - \rho V_t dt\}. \tag{8}$$

4. EXPLICIT SOLUTIONS AND COMPARATIVE STATICS

In general, it is difficult to work out explicit solutions of the functional equation 8 and get a simple formula of the unutilized patent. However, analysis is reduced to the case of Nishimura and Ozaki (2007) if (i) the profit process follows geometric Brownian motion; (ii) $U(x) = x$; (iii) the planning horizon is infinite. In this paper, we change the assumption of (ii) by (ii)′ $U(x) = \ln x$ for describing the risk averse of the DM. Now, we are in position to explicitly solve the optimal stopping problem in such a case and get a simple explicit formula of the unutilized patent.

4.1. Explicit solution with logarithmic utility

As described in the above assumptions, we assume that the operating profit process (π_t) is generated by a geometric Brownian motion [4]:

$$d\pi_t = \pi_t(\mu dt + \sigma dB_t), \quad \pi_0 > 0,$$

where μ and $\sigma > 0$ are two constants. Girsanov's theorem implies that

$$d\pi_t = \pi_t((\mu - \sigma\theta_t)dt + \sigma dB_t^\theta)$$

[4] In this case, if we assume $k = 0, t = 0$ and $U(x) = x$, in view of 7, then we may get the optimal stopping problem of the DM can be rewritten as

$$\max_\tau \mathbb{E}[\int_\tau^T e^{-\rho t}\pi_t dt] = \int_\tau^T \exp{(\mu - \rho)t} dt,$$

thus, the value of the project could be made indefinitely larger by choosing a larger τ as T approaching ∞, and no optimum would exist. This is the explanation of $\rho > \mu$.

where $B_t^\theta = B_t + \int_0^t \theta_s ds$ is a Brownian motion under the probability measure $\mathbb{Q}^\theta$. This and Ito formula yield

$$\pi_t = \pi_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t - \sigma \int_0^t \theta_s ds + \sigma B_t^\theta\}.$$

Here, we chose the logarithmic utility, which implies that the DM has decreasing absolute risk aversion, that is, as its wealth approaches zero, the DM becomes infinitely risk averse.

In the discussion that follows, we will split our discussion into two steps in order. The first one is to derive the explicit of the utilized patent, and the second is to work out the explicit formula of the optimal stopping problem.

Firstly, in this case, 6 can be rewritten as

$$W(\pi_t, t) = \int_t^T e^{-\rho(s-t)} \mathbb{E}_{\mathbb{Q}^k}\left[\ln\left(\pi_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t - \sigma \int_0^t k ds + \sigma B_t^k\}\right) |\mathcal{F}_t\right] ds,$$

simple calculations (see Appendix) yield

$$W(\pi_t, t) = \ln \pi_t \cdot \frac{1}{\rho}(1 - e^{-\rho(T-t)}) - \frac{\mu - \sigma k - \frac{1}{2}\sigma^2}{\rho}(\frac{\rho(T-t) - 1}{\rho e^{\rho(T-t)}} + \frac{1}{\rho}). \quad (9)$$

For simplicity, we assume that the time horizon T approaches infinity [5], thus

$$\lim_{T\to\infty} e^{-\rho(T-t)} = 0,$$

and

$$\lim_{T\to\infty} \frac{\rho(T-t) - 1}{\rho e^{\rho(T-t)}} = \lim_{T\to\infty} \frac{\rho}{\rho^2 e^{\rho(T-t)}} = 0,$$

which and 9 imply that $W(\pi_t, t)$ depends only on π_t, i.e.

$$W(\pi_t) = \frac{\ln \pi_t}{\rho} - \frac{\mu - \sigma k - \frac{1}{2}\sigma^2}{\rho^2}. \quad (10)$$

In view of the expression of π_t and 10, Ito's Lemma yields (see Appendix)

$$dW_t = \frac{1}{\rho}((\mu - \sigma\theta_t - \frac{1}{2}\sigma^2)dt + \sigma dB_t^\theta) \quad (11)$$

[5]In what follows, when T goes infinity, we assume the relations between variables in the limit also hold in infinite case. For proving these result, we need some more mathematical techniques, here we omit it. For instance, the Girsanov's theorem of infinite horizon case can be found in Karatzas and Shreve (1991).

where $W_0 = \frac{\ln \pi_0}{\rho} - \frac{\mu - \sigma k - \frac{1}{2}\sigma^2}{\rho^2}$.

Now, turn to the second step, we are in position to explicitly solve the optimal stopping problem. In this case, if the planning horizon is infinite and W_t follows 11, then V_t defined by 8 depends only on W_t, and not on physical time t. Therefore, we are allowed to rewrite it as $V_t = V(W_t)$ with some $V : \mathbb{R} \to \mathbb{R}$. So the Hamilton-Jacobi-Bellman equation turns out to be

$$V(W_t) = \max\{W_t - I, \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[dV_t|\mathcal{F}_t] + V_t - \rho V_t dt\}. \quad (12)$$

To solve the above HJB equation. We conjecture that there exists W^*[6] such that the optimal strategy of the DM takes the form of "stop right now" if $W_t \geq W^*$ and waits if $W_t < W^*$. This conjecture will be verified to be true later.

In the continuation region, that is, when $W_t < W^*$, it holds from 12 and 11 that [7]

$$\begin{aligned}
\rho V(W_t)dt &= \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[dV_t|\mathcal{F}_t] \\
&= \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[V'(W_t)\frac{1}{\rho}((\mu - \sigma\theta_t - \frac{1}{2}\sigma^2)dt + \sigma dB_t^\theta) \\
&\qquad + \frac{1}{2}\frac{\sigma^2}{\rho^2}V''(W_t)dt|\mathcal{F}_t] \\
&= \min_\theta V'(W_t)\frac{1}{\rho}(\mu - \sigma\theta_t - \frac{1}{2}\sigma^2)dt + \frac{1}{2}\frac{\sigma^2}{\rho^2}V''(W_t)dt \\
&= V'(W_t)\frac{1}{\rho}(\mu - \sigma k - \frac{1}{2}\sigma^2)dt + \frac{1}{2}\frac{\sigma^2}{\rho^2}V''(W_t)dt.
\end{aligned}$$

Here, we also conjecture that V is twice differentiable in the continuation region, and V' is positive. All these imply that $V(\cdot)$ satisfies the following ordinary differential equation (ODE)

$$\sigma^2 V''(W_t) + 2\rho(\mu - \sigma k - \frac{1}{2}\sigma^2)V'(W_t) - 2\rho^3 V(W_t) = 0, \quad (13)$$

with boundary conditions $V(0) = 0, V(W^*) = W^* - I$ and $V'(W^*) = 1$. If the utilized patent has no value, then the investment opportunity also has no value, which means that the boundary condition $V(0) = 0$ holds; from the expression of 12, we have $V(W^*) = W^* - I$, which implies $V'(W^*) = 1$.

[6]To exclude the trivial case, we assume that $W^* \geq \frac{\ln I}{\rho}$, which can be obtained by $W(I, t) = \int_t^T e^{-\rho(s-t)} \ln I ds$ in view of 6 and the approaching infinity of T.

[7]Here, we use the conjecture that $V'(\cdot)$ is positive in the fourth equality.

Solving equation 13 (see Appendix), we obtain

$$V(W_t) = C(e^{\alpha W_t} - e^{\beta W_t}), \tag{14}$$

where

$$\alpha = \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) + \rho\sqrt{(\mu - \sigma k - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} \tag{15}$$

$$\beta = \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) - \rho\sqrt{(\mu - \sigma k - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} \tag{16}$$

$$C = \frac{1}{\alpha e^{\alpha W^*} - \beta e^{\beta W^*}} \tag{17}$$

and

$$W^* = \max\left\{\frac{\ln I}{\rho}, \frac{\ln(\frac{\beta}{\alpha})^2}{\alpha - \beta}\right\}. \tag{18}$$

Summarying the above results, we conclude that the value of the investment opportunity or the patent V is

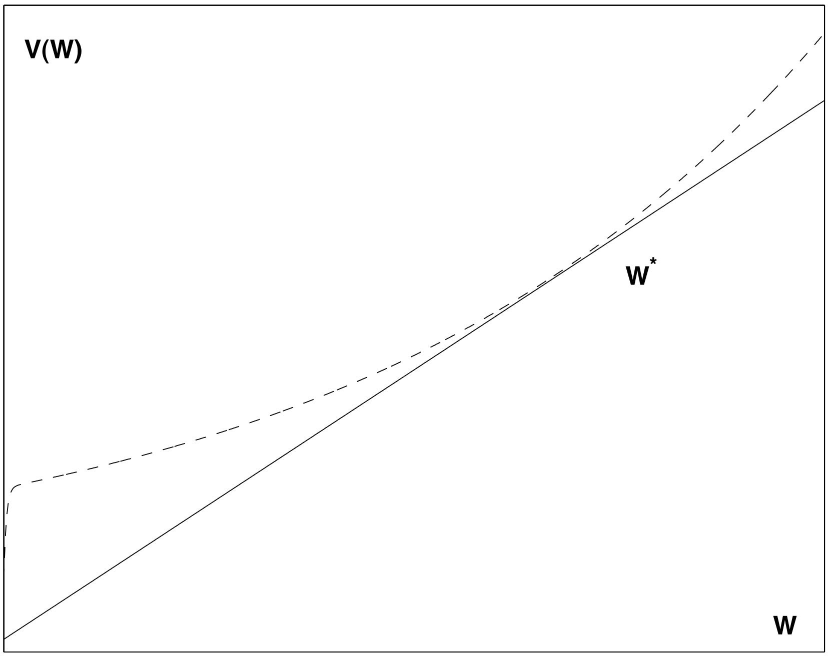
$$V(W_t) = \begin{cases} C(e^{\alpha W_t} - e^{\beta W_t}), \text{ if } W_t < W^* \\ W^* - I, \text{ if } W_t \geq W^*. \end{cases} \tag{19}$$

Recall that we have made three conjectures: (i) There exists a reservation value W^*; (ii) V is twice differentiable in the continuation; and (iii) V' is positive. (ii) is simple in view of the expression 19; (iii) $V'(W_t) = C(\alpha e^{\alpha W_t} - \beta e^{\beta W_t}) \geq 0$ because of $C \geq 0, \alpha \geq 0$ and $\beta \leq 0$; and the Figure 1 illustrates that (i) holds, where the solid and the dashed curve represent the function of $W - I$ and $C(e^{\alpha W} - e^{\beta W})$, respectively.

Summarying all the results above, we claim the following proposition.

PROPOSITION 3. *Suppose that the DM is risk- and uncertainty-averse under k-ignorance with logarithmic utility, the profit process follows geometric Brownian motion, and further assume that relations among variables in the finite-horizon case converges, as the horizon goes to infinity, to those in the infinite-horizon case. Then, in the case of infinite horizon, the value of the unutilized patent, that is, $V(W_t)$ in the continuation region, is given by 14 with α, β, C and W^* defined by 15, 16 17 and 18, respectively.*

FIG. 1. The Existence of Reservation Value of W^*



4.2. Comparative static

In the discussion that follows, we are in position to analyze the effect of the risk (σ) and the uncertainty (k) on the value of the utilized patent, the value of the unutilized patent and the value of waiting. Since most of the interrelations are rather complex, we refer to numerical examples in order to demonstrate the different effects. In the numerical illustrations presented below, we fix the investment parameters, unless otherwise stated, as follows: $\rho = 0.1, \mu = 0.05, I = 1, \sigma = 0.2, k = 0.3$ and $\pi_t = e$.

- The value of the utilized patent

In view of the expression 10, we conclude that

$$W(\pi_t) = \frac{\rho \ln \pi_t - \mu + \sigma k + \frac{1}{2}\sigma^2}{\rho^2}.$$

Consequently, an increasing in k will increase the value of the utilized patent, furthermore, an increasing in σ, even if there is no uncertainty ($k = 0$), will increase the value of the utilized patent. All these results mean that the high risk and the high uncertainty will bring the high payoff, which is consistent with the classical result and the intuition.

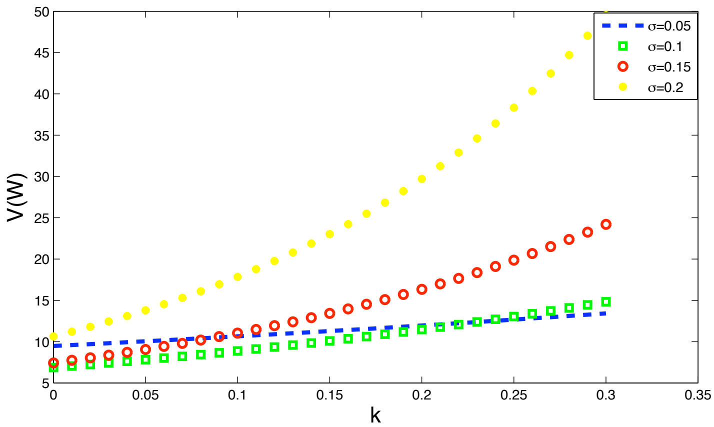
- The value of the unutilized patent

Let us turn to the case of the unutilized patent, i.e. $W_t \leq W^*$, then

$$V(W_t) = C(e^{\alpha W_t} - e^{\beta W_t})$$

where $C = \frac{1}{\alpha e^{\alpha W^*} - \beta e^{\beta W^*}}$ and $W^* = \max\left\{\frac{\ln I}{\rho}, \frac{\ln(\frac{\beta}{\alpha})^2}{\alpha-\beta}\right\}$. Here, without loss of generality, we consider the cases of $\sigma = 0.05, 0.1, 0.15, 0.2$ and $k = 0, 0.1, 0.2, 0.3$. Figure 2 and Figure 3 listed below express that the value of the unutilized patent increases with the increase of the risk and the uncertainty. This and the effect on the value of the utilized patent imply that the value of the utilized patent and the unutilized patent will increase with the increase of risk and uncertainty.

FIG. 2. The Value of the Unutilized Patent Varies with the Varying of Risk



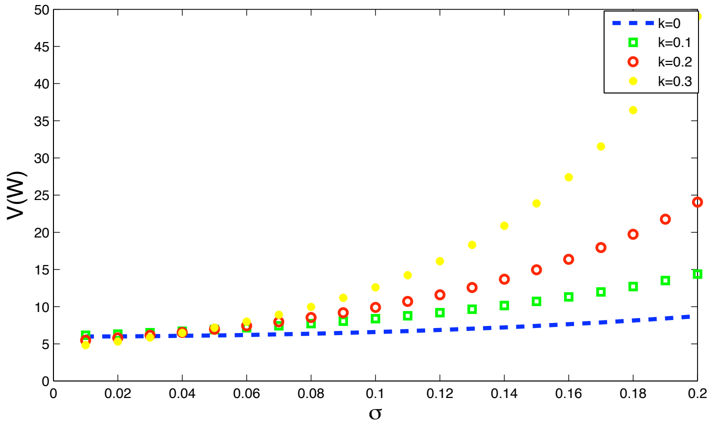
- The value of the reservation profit flow

Now, we are in position to analyze the relationship between the investment timing and the risk, the uncertainty. Since the time of the investment is fully determined by the instant current profit π_t exceed the threshold level π^* for the first time. All that matters is to examine the effect of the risk and the uncertainty on reservation profit flow. Expression 10 implies that

$$\pi^* = \exp\{\rho W^* + \frac{\mu - \sigma k - \frac{1}{2}\sigma^2}{\rho}\},$$

which and $W^* = \max\left\{\frac{\ln I}{\rho}, \frac{\ln(\frac{\beta}{\alpha})^2}{\alpha-\beta}\right\}$ mean that the reservation profit flow has the following relation with the risk and the uncertainty. Here, without

FIG. 3. The Value of the Unutilized Patent Varies with the Varying of Uncertainty



loss of generality, we also consider the case of $\sigma = 0.05, 0.1, 0.15, 0.2$ and $k = 0, 0.1, 0.2, 0.3$. Firstly, we analyze the effect of the uncertainty on the reservation profit flow, the conclusion is that the reservation profit flow decreases with the increase of the uncertainty, See Figure 4. Secondly, Figure 5 implies that the reservation profit flow increases with the increase of the risk when $\sigma \leq 0.12$ and $k = 0, 0.1$. However, after that, the reservation profit flow decreases with the increase of the risk (Figure 6). To sum up, when the risk and the uncertainty are small, they have a positive impact on the reservation profit flow, which means that the DM can bear this small risk and uncertainty; on the other hand, when the risk and the uncertainty are large, they have a negative impact on the reservation profit flow, which implies that the DM is risk- and uncertainty-averse. All these results are consistent with our intuition.

5. CONCLUSION

In this paper, the approach of BSDE is employed to study the irreversible investment problem under k-ignorance when the DM is risk- and uncertainty-averse. The economic motivation is that the DM is not only uncertainty averse but also risk averse, and the mathematical innovation is the use of BSDE made the irreversible investment problem more realistic and the derivations more simple. For the case of logarithmic utility and the geometric Brownian motion profit flow process, we obtain the explicit expressions of the value of the utilized patent, the value of the unutilized patent and the value of the reservation profit flow. Following this, we analyze the effects of the risk and the uncertainty on the above three pa-

FIG. 4. The Value of the Reservation Profit Flow Varies with the Varying of Uncertainty

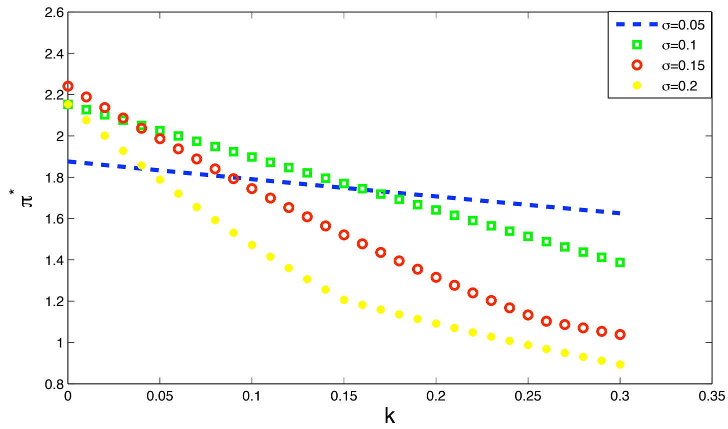
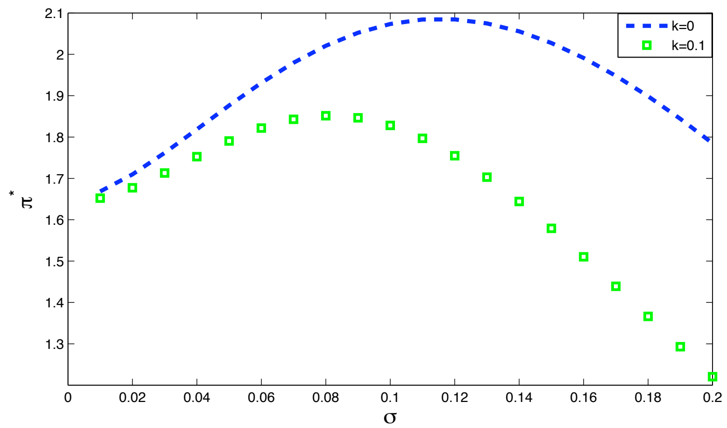
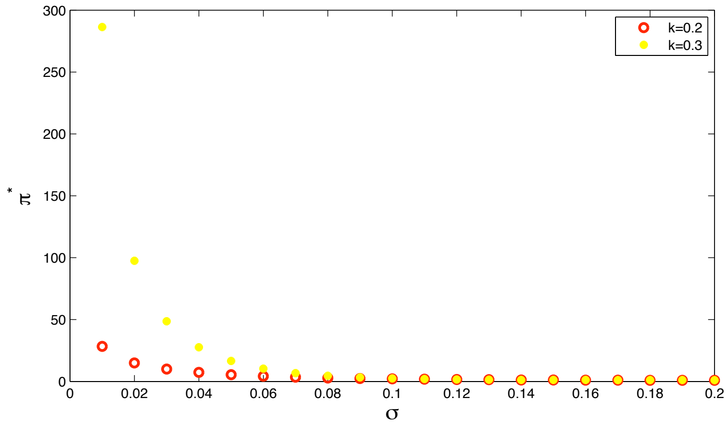


FIG. 5. The Value of the Reservation Profit Flow Varies with the Varying of Risk ($k = 0, k = 0.1$)



rameters by numerical illustrations. The conclusion is that the risk and the uncertainty have the positive impact on the value of the utilized patent and the unutilized patent. For the small risk and uncertainty, they also have the positive impact on the value of the reservation profit flow. All these results are consistent with our intuition.

FIG. 6. The Value of the Reservation Profit Flow Varies with the Varying of Risk ($k = 0.2, k = 0.3$)



APPENDIX A

The proof of Lemma 4. In view of Lemma 1, BSDE 2 has a unique solution, say (y_t, z_t). Put

$$a_t = k\mathrm{sgn}(z_t) \text{ and } \frac{d\mathbb{Q}^a}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^T |a_s|^2 ds - \int_0^T a_s dB_s\right\},$$

then $\sup_{0 \leq t \leq T} |a_t| \leq k, \mathbb{Q}^a \in \mathcal{P}$. In this case, Girsanov's theorem yields that $\bar{B}_t = B_t + \int_0^t a_s ds$ is a Brownian motion under probability measure $\mathbb{Q}^a$, which implies that BSDE 2 can be rewritten as

$$Y_t = \xi - \int_t^T z_s d\bar{B}_s, 0 \leq t \leq T. \tag{A.1}$$

Taking the conditional expectation $\mathbb{E}_{\mathbb{Q}^a}[\cdot|\mathcal{F}_t]$ on both side of BSDE A.1, we obtain

$$y_t = \mathbb{E}_{\mathbb{Q}^a}[\xi|\mathcal{F}_t] \geq \inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t] = Y_t.$$

On the other hand, let $\{h_t\}$ be an adapted process bounded by k, that is $\sup_{0 \leq t \leq T} |h_t| \leq k$. Again by Lemma 1, the following BSDE has a unique solution (y^h, z^h),

$$y_t^h = \xi - \int_t^T h_s |z_s^h| ds - \int_t^T z_s^h dB_s, 0 \leq t \leq T. \tag{A.2}$$

Let $\frac{d\mathbb{Q}^h}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^T |h_s|^2 ds - \int_0^T h_s dB_s\right\}$, then solving BSDE A.2 yields

$$y_t^h = \mathbb{E}_{\mathbb{Q}^h}[\xi|\mathcal{F}_t].$$

Note that $-h_t z \geq -k|z|$, which and comparison theorem of BSDE (ref. El Karoui et al., 1997) imply

$$\mathbb{E}_{\mathbb{Q}^h}[\xi|\mathcal{F}_t] = y_t^h \geq y_t,$$

thus

$$Y_t = \inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t] \geq y_t.$$

Summarying all the results above, we have

$$y_t = Y_t, \quad t \in [0, T],$$

which means that $\inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\xi|\mathcal{F}_t]$ is the solution of BSDE 2. ▮

The proof of Proposition 1. Fubini theorem of conditional expectation and Lemma 5 imply

$$\begin{aligned} W(\pi_t, t) &= \inf_{\mathbb{Q}^\theta \in \mathcal{P}} \int_t^T e^{-\ \rho(s-t)} \mathbb{E}_{\mathbb{Q}^\theta}[U(\pi_s)|\mathcal{F}_t] ds \\ &= \int_t^T e^{-\rho(s-t)} \inf_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[U(\pi_s)|\mathcal{F}_t] ds \\ &= \int_t^T e^{-\rho(s-t)} \mathbb{E}_{\mathbb{Q}^k}[U(\pi_s)|\mathcal{F}_t] ds. \end{aligned}$$

This is the desired result. ▮

The proof of Proposition 2. In this case, we have

$$\begin{aligned}
V_t &= \max_{\tau \geq t} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\int_\tau^T e^{-\rho(s-t)} U(\pi_s) ds - e^{-\rho(\tau - t)} I | \mathcal{F}_t] \\
&= \max\{ \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\int_t^T e^{-\rho(s-t)} U(\pi_s) ds | \mathcal{F}_t] - I, \\
&\quad \max_{\tau \geq t+dt} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\int_\tau^T e^{-\rho(s-t)} U(\pi_s) ds - e^{-\rho(\tau - t)} I | \mathcal{F}_t]\} \\
&= \max\{W_t - I, e^{-\rho dt} \max_{\tau \geq t+dt} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\int_\tau^T e^{-\rho(s-t-dt)} U(\pi_s) ds - e^{-\rho(\tau - t-dt)} I | \mathcal{F}_t]\} \\
&= \max\{W_t - I, \\
&\quad e^{-\rho dt} \max_{\tau \geq t+dt} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\min_{\mathbb{Q}^{\theta'} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^{\theta'}}[\int_\tau^T e^{-\rho(s-t-dt)} U(\pi_s) ds - e^{-\rho(\tau - t-dt)} I | \mathcal{F}_{t+dt}] | \mathcal{F}_t]\} \\
&= \max\{W_t - I, \\
&\quad e^{-\rho dt} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[\max_{\tau \geq t+dt} \min_{\mathbb{Q}^{\theta'} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^{\theta'}}[\int_\tau^T e^{-\rho(s-t-dt)} U(\pi_s) ds - e^{-\rho(\tau - t-dt)} I | \mathcal{F}_{t+dt}] | \mathcal{F}_t]\} \\
&= \max\{W_t - I, e^{-\rho dt} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[V_{t+dt} | \mathcal{F}_t]\} \\
&= \max\{W_t - I, e^{-\rho dt} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[dV_t | \mathcal{F}_t] + V_t\} \\
&= \max\{W_t - I, (1 - \rho dt)(\min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[dV_t | \mathcal{F}_t] + V_t)\} \\
&= \max\{W_t - I, \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}^\theta}[dV_t | \mathcal{F}_t] + V_t - \rho V_t dt\}
\end{aligned}$$

where each equality holds by: splitting the decision between investing now (at time t) and waiting for a short time interval and reconsidering whether to invest or not after it (at time $t + dt$)(second); Remark 2.2 and (iii) in Lemma 3 (fourth); the definition of V_t, with t replaced by $t+dt$(sixth); writing V_{t+dt} as V_t+dV_t (seventh); approximating $e^{-\rho dt}$ by $(1-\rho dt)$ (such an approximation is justified since we let dt go to zero) (eighth); and eliminating the term which is of a higher order than dt (ninth). ▮

Derivation of 9. In this case, we have

$$\begin{aligned}
W(\pi_t, t) &= \int_t^T e^{-\rho(s-t)} \mathbb{E}_{\mathbb{Q}^k}[\ln \pi_0 + (\mu - \sigma k - \frac{1}{2}\sigma^2)s + \sigma B_s^k | \mathcal{F}_t] ds \\
&= \int_t^T e^{-\rho(s-t)} (\ln \pi_0 + (\mu - \sigma k - \frac{1}{2}\sigma^2)s + \sigma B_t^k) ds \\
&= \int_t^T e^{-\rho(s-t)} (\ln \pi_0 + (\mu - \sigma k - \frac{1}{2}\sigma^2)s + \sigma B_t^k) ds \\
&= \int_t^T e^{-\rho(s-t)} (\ln \pi_t + (\mu - \sigma k - \frac{1}{2}\sigma^2)(s-t)) ds \\
&= \int_t^T e^{-\rho(s-t)} \ln \pi_t + \int_t^T (\mu - \sigma k - \frac{1}{2}\sigma^2)(s-t) e^{-\rho(s-t)} ds
\end{aligned}$$

where

$$\begin{aligned}\int_t^T \ln\pi_t \cdot e^{-\rho(s-t)}ds &= \ln\pi_t \cdot -\frac{1}{\rho}(e^{-\rho(T-t)}-1)\\ &= \ln\pi_t \cdot \frac{1}{\rho}(1-e^{-\rho(T-t)}),\end{aligned}$$

and

$$\begin{aligned}&\int_t^T (\mu-\sigma k-\frac{1}{2}\sigma^2)(s-t)e^{-\rho(s-t)}ds\\ = & -\frac{1}{\rho}\int_t^T (\mu-\sigma k-\frac{1}{2}\sigma^2)(s-t)de^{-\rho(s-t)}\\ = & -\frac{1}{\rho}(\mu-\sigma k-\frac{1}{2}\sigma^2)(s-t)e^{-\rho(s-t)}|_t^T\\ + & \frac{1}{\rho}\int_t^T (\mu-\frac{1}{2}\sigma^2-k\sigma)e^{-\rho(s-t)}ds\\ = & -\frac{\mu-\sigma k-\frac{1}{2}\sigma^2}{\rho}(T-t)e^{-\rho(T-t)}-\frac{\mu-\sigma k-\frac{1}{2}\sigma^2}{\rho^2}(e^{-\rho(T-t)}-1)\\ = & -\frac{\mu-\sigma k-\frac{1}{2}\sigma^2}{\rho}((T-t)e^{-\rho(T-t)}-\frac{1}{\rho}e^{-\rho(T-t)}+\frac{1}{\rho})\\ = & -\frac{\mu-\sigma k-\frac{1}{2}\sigma^2}{\rho}(\frac{\rho(T-t)-1}{\rho e^{\rho(T-t)}}+\frac{1}{\rho}),\end{aligned}$$

which imply

$$W(\pi,t)=\ln\pi_t\cdot\frac{1}{\rho}(1-e^{-\rho(T-t)})-\frac{\mu-\sigma k-\frac{1}{2}\sigma^2}{\rho}(\frac{\rho(T-t)-1}{\rho e^{\rho(T-t)}}+\frac{1}{\rho}).$$

This the desired result. ▮

Derivation of 11. Itô formula and

$$W(\pi_t)=\frac{\ln\pi_t}{\rho}-\frac{\mu-k\sigma-\frac{1}{2}\sigma^2}{\rho^2}$$

imply

$$\begin{aligned}dW_t &= \frac{1}{\rho}\frac{1}{\pi_t}d\pi_t-\frac{1}{2}\frac{1}{\rho}\frac{1}{\pi_t^2}d\langle\pi\rangle_t\\ &= \frac{1}{\rho}(\mu dt+\sigma dB_t)-\frac{1}{2\rho}\sigma^2 dt\\ &= \frac{1}{\rho}((\mu-\sigma\theta_t-\frac{1}{2}\sigma^2)dt+\sigma dB_t^\theta).\end{aligned}$$

This is the desired result. ▮

Solving ODE 11. In view of preliminary results of ODE, we have

$$V(W_t) = C_1 e^{\alpha W_t} + C_2 e^{\beta W_t},$$

where

$$\begin{aligned}\alpha &= \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) + \rho\sqrt{(\mu - \sigma k - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} \\ &\geq \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) + \rho|\mu - \sigma k - \frac{1}{2}\sigma^2|}{\sigma^2} \geq 0,\end{aligned}$$

and

$$\begin{aligned}\beta &= \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) - \rho\sqrt{(\mu - \sigma k - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} \\ &\leq \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) - \rho|\mu - \sigma k - \frac{1}{2}\sigma^2|}{\sigma^2} \leq 0.\end{aligned}$$

Furthermore, three boundary conditions imply that W^*, C_1 and C_2 satisfy the following equation

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 e^{\alpha W^*} + C_2 e^{\beta W^*} = W^* - I \\ C_1 \alpha e^{\alpha W^*} + C_2 \beta e^{\beta W^*} = 1. \end{cases}$$

Equation $C_1 + C_2 = 0$ means that $C_1 = -C_2$, we denoted C_1 by C. $C(\alpha e^{\alpha W^*} - \beta e^{\beta W^*}) = 1, \alpha \geq 0$ and $\beta \leq 0$ imply that $C = \frac{1}{\alpha e^{\alpha W^*} - \beta e^{\beta W^*}} \geq 0$. Thus, W^* satisfies the following equation

$$\frac{e^{\alpha W^*} - e^{\beta W^*}}{\alpha e^{\alpha W^*} - \beta e^{\beta W^*}} = W^* - I,$$

which implies that

$$e^{\alpha W^*} - e^{\beta W^*} = (W^* - I)(\alpha e^{\alpha W^*} - \beta e^{\beta W^*}),$$

differentiating the above equation w.r.t. W^*, we have

$$\alpha e^{\alpha W^*} - \beta e^{\beta W^*} = \alpha e^{\alpha W^*} - \beta e^{\beta W^*} + (W^* - I)(\alpha^2 e^{\alpha W^*} - \beta^2 e^{\beta W^*}).$$

Thus,

$$(W^* - I)(\alpha^2 e^{\alpha W^*} - \beta^2 e^{\beta W^*}) = 0,$$

which and hypothesis $W^* \geq \frac{\ln I}{\rho}$ imply that

$$W^* = \max\left\{\frac{\ln I}{\rho}, \frac{\ln(\frac{\beta}{\alpha})^2}{\alpha - \beta}\right\},$$

To sum up, we conclude that V has the following form

$$V(W_t) = C(e^{\alpha W_t} - e^{\beta W_t})$$

where

$$\begin{aligned}
\alpha &= \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) + \rho\sqrt{(\mu - \sigma k - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} \\
\beta &= \frac{-\rho(\mu - \sigma k - \frac{1}{2}\sigma^2) - \rho\sqrt{(\mu - \sigma k - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} \\
C &= \frac{1}{\alpha e^{\alpha W^*} - \beta e^{\beta W^*}},
\end{aligned}$$

and

$$W^* = \max\left\{\frac{\ln I}{\rho}, \frac{\ln \frac{\beta}{\alpha}}{\alpha - \beta}\right\}.$$

▮

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